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On Nonlinear Stationary Half-Space Problems in Discrete Kinetic Theory

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We show that every steady discrete velocity model of the Boltzmann equation on the real line, $\xi_i \cdot (d/dx) f^i = C^i(f)$, which satisfies an H-theorem and for which all $\xi_i \neq 0$, has solutions on the half-line $(0, \infty)$ which take prescribed nonnegative $f^i(0)$ if $\xi_i > 0$ and approach a certain manifold of Maxwellians as $x \to \infty$. Such solutions give the density distribution in a Knudsen boundary layer in the discrete velocity case.

KEY WORDS: Boltzmann equation; steady discrete velocity models; half-space problem.

1. INTRODUCTION

The importance of half-space problems for the Boltzmann equation stems from their role as boundary layer solutions associated with more complicated situations where a significant change in the boundary data occurs in a distance on the order of a mean free path.[13] Such changes are important in the kinetic boundary layers, which are different from the usual viscous boundary layers and have the thickness of a few mean free paths; these layers are sometimes called the Knudsen layers.[6,7] Formally, it can be seen that if the boundary of a domain is flat or if it has a radius of curvature much larger than the mean free path, then the behavior of the distribution in the Knudsen layer can be found by solving the Boltzmann equation in a half-space.

The first appearance of half-space problems in the kinetic theory of gases of classical molecules, the ones Boltzmann had in mind when he derived his equation, dates back to 1949 and is due to Kramers.[20] A

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systematic analysis of perturbation procedures led Grad[18] to recognize the existence and importance of the Knudsen layers. The need for such solutions seems to have been most urgent in problems of linear transport, such as those occurring in stellar or planetary atmospheres[14] and in neutron transport.[16] Such linear problems have been studied extensively.[1,2,4,6–11,19] Specifically, Beals[4] and Greenberg and van der Mee[19] made use of operator techniques to work on abstract versions of the linearized Boltzmann equation, while Bardos *et al.*[2] adopted a more elementary and direct approach in the case of hard spheres. Their treatment was extended by Cercignani to hard cutoff potentials.[11] A nonlinear treatment of a perturbation from equilibrium has been announced.[3]

In this paper, the existence question for fully nonlinear half-space problems is studied in the simpler case of discrete velocity models, in order to avoid a technical difficulty connected with small velocity components perpendicular to the plane bounding the half-space. The rationale for this is given in the study[12] of such models for boundary value problems in a slab.

The plan of the paper is as follows. In Section 2, we introduce the stationary discrete velocity models for the one-dimensional geometry which is relevant in the half-space situation, and we formulate the problem. The notation for the density and the invariant density, momentum, and energy fluxes is given, and an H-theorem is discussed. Section 3 contains a short presentation and discussion of the Maxwellian equilibrium solutions in the continuous case. An auxiliary result for a boundary value problem in a slab is proved in Section 4. The main theorem is formulated and proved in Section 5.

2. THE MODELS, THE PROBLEM, AND SOME NOTATION

We are concerned with discrete velocity models of the Boltzmann equation. The particles can only have one of finitely many velocities $v_1, \ldots, v_n \in \mathbb{R}^3$. The x, y, and z components of v_i are denoted by ξ_i, η_i, and ζ_i. The particles moving with velocity v_i at time t and at the space point $r = (x, y, z)$ are described by a density distribution function f^i evaluated at (t, x, y, z). In the steady half-space case, the f_i depend only on x and satisfy the system of equations

$$\xi_i \cdot \frac{d}{dx} f^i = C^i(f), \qquad i = 1, \ldots, n \tag{2.1}$$

Here, $f = (f^1, \ldots, f^n)$ is the vector function given by all the densities, and the collision operator $C^i(f)$ has the form

$$C^i(f) = \sum_{j,k,l} A^{ij}_{kl}(f^k f^l - f^i f^j) \tag{2.2}$$

The transition rates A_{kl}^{ij} are assumed to satisfy the usual identities $A_{kl}^{ij} = A_{lk}^{ij} = A_{lk}^{ji} = A_{ij}^{kl}$ for all i, j, k, l. In particular, the model satisfies an H-theorem. Our basic assumption in this paper is that $\xi_i \neq 0$ for all $i = 1,..., n$. Given this assumption, we look for nonnegative solutions f of (2.1) on $\mathbb{R}_+$ satisfying the following boundary conditions:

For $\xi_i > 0$, we prescribe $f^i(0) = \alpha_i \geqslant 0$ (2.3)

We assume that at least one α_i is strictly positive

Let $m = (m^1,..., m^n)$ be a (Maxwellian) equilibrium solution of (2.1); then we require that for some such m,

$$\lim_{x \to \infty} f^i(x) = m^i \tag{2.4}$$

The motivation for the boundary condition (2.4) is that at the boundary between the Knudsen and the viscous boundary layer, the gas should be in equilibrium, and only fluid dynamics is necessary to deal with the viscous layer.

Here m is a Maxwellian equilibrium of (2.1) if and only if $C^i(m) = 0$ for all i. As in classical kinetic theory, a Maxwellian state is completely determined by its summational invariants,[5,17] which, for the full Boltzmann equation, are just mass, momentum, and energy. We assume that the discrete models considered here are all such that mass, momentum, and energy are conserved in the time-dependent case, i.e., the collision terms $C^i(f)$ must satisfy

$$\sum_i C^i(f) = 0 \tag{2.5}$$

$$\sum_i v_i C^i(f) = 0 \tag{2.6}$$

$$\sum_i v_i^2 C^i(f) = 0 \tag{2.7}$$

Note that (2.6) is a vector identity. Depending on the particular model under consideration, there are sometimes additional collision invariants (see ref. 17 for a general discussion).

Equations (2.5)–(2.7) imply that the fluxes

$$j := j(f) := \sum_i \xi_i f^i(x) \tag{2.8}$$

$$p := p(f) := \sum_i \xi_i v_i f^i(x) \tag{2.9}$$

$$e := e(f) := \sum_i \xi_i v_i^2 f^i(x) \tag{2.10}$$

are independent of x. Here p is of course a vector, and we denote its components by (p_1, p_2, p_3). For j, p, and e fixed, we denote by M the manifold of all Maxwellians having these fluxes. Note that M may consist of only a finite number of points; this is true, for example, in the continuous case when $j > 0$ (see Section 3).

Frequently we consider quantities which involve only velocities whose x component has just one sign. For example, $\rho^+(x) := \sum_i^+ f^i(x)$ is defined as the sum over all particle densities for which $\xi_i > 0$. Analogously, we define $\rho^-(x)$, $j^+(x)$, $p_1^+(x)$, etc. For $j^-(x)$, we choose the definition $j^-(x) = \sum_i^- |\xi_i| f^i(x)$, i.e., we have

$$j(x) = j^+(x) - j^-(x)$$

There is an analog to the Boltzmann H-theorem in the steady case. Let $h(y) = y \ln y$ for $y > 0$, and $h(y) = 0$ for $y = 0$, and define the "H-flux" (or "negative entropy flux") by

$$\tilde{H}[f](x) = \sum_i \xi_i h(f^i(x)) \tag{2.11}$$

If f is a solution of (2.1), it follows as usual that

$$\frac{d}{dx} \tilde{H}[f](x) \leqslant 0$$

with equality at some x if and only if $f^i f^j(x) = f^l f^k(x)$ for all i, j, k, l for which $A_{kl}^{ij} \neq 0$, i.e., if and only if f is a Maxwellian at x.

Our analysis will be done in the space C_b of bounded continuous functions on $[0, \infty)$. Let $C_{b,+}$ denote the cone of positive functions in C_b, and $(C_b)^n$ the n-fold Cartesian product of C_b with itself.

3. MAXWELLIANS

The quantities j, p, and e are invariants of Eq. (2.1). However, the data at the wall $x = 0$ are insufficient to predict the values of j, p, and e. Two questions arise naturally in connection with our problem: Can we prescribe the values of j, p, and e (or at least some of them)? And do the values of j, p, and e determine the Maxwellian uniquely?

In this section, we give a partial answer to the first question, and a complete answer to the second question in the case of the full Boltzmann equation.

An elementary estimate shows that it is impossible to prescribe any value of j at infinity. In fact, for any nonnegative solution f we must have

$$j(m) = j(f) = j^+(f)(0) - j^-(f)(0) \leqslant j^+(f)(0) \tag{3.1}$$

and $j^+(f)(0)$ is given. This shows that $j(m)$ must certainly satisfy the constraint given by (3.1). Which other constraints j, p, and e must satisfy in detail to be admissible as fluxes of the Maxwellian at infinity is a question we are unable to answer.

We now discuss question 2 for the Boltzmann equation. Let

$$m_{u,\beta,\rho}(v) = \rho(\pi/\beta)^{-3/2} \exp[-\beta(v-u)^2]$$

be the Maxwellian in $\mathbb{R}^3$ with density ρ and mean velocity u.

If j, p, and e are given, an easy calculation shows that ρ, β, and u must satisfy the equations

$$\rho u_1 = j \tag{3.2}$$

$$\rho(1/2\beta + u_1^2) = p_1 \tag{3.3}$$

$$\rho(u_1 u_2) = p_2 \tag{3.4}$$

$$\rho(u_1 u_3) = p_3 \tag{3.5}$$

$$\rho u_1(5/2\beta + u^2) = e \tag{3.6}$$

We distinguish the two cases $j=0$ and $j>0$. For the case $j=0$, we must necessarily have $u_1=0$ (or $\rho=0$, but this leads immediately to vacuum). Equations (3.2)–(3.6) are then only solvable if also $p_2=p_3=e=0$, but then u_2 and u_3 can be chosen arbitrarily. Equation (3.3) now reads $\rho = 2p_1\beta$. For $p_1>0$, this establishes a relation between ρ and β.

We summarize the result: For $j=p_2=p_3=e=0$, we have a manifold of Maxwellians, determined by the parameters $\beta>0$, u_2, and u_3 and by the relations $\rho=2p_1\beta$ and $u_1=0$. In particular, there are infinitely many Maxwellians for every $p_1>0$.

The entropy flux $\tilde{H}[f](x) = \int \xi f(x,v) \ln f(x,v)\, dx$ is easily seen to vanish identically on all these Maxwellians.

The case $j>0$ is less degenerate: We easily obtain

$$u_2 = p_2/j, \qquad u_3 = p_3/j \tag{3.7}$$

$$\beta = \rho^2/2(\rho p_1 - j^2) \tag{3.8}$$

For the density ρ we solve a quadratic equation, which has at most two real solutions

$$\rho = \frac{5p_1 \pm \{25p_1^2 - 16[ej - (p_2^2+p_3^2)]\}^{1/2}}{2[e/j - (p_2^2+p_3^2)/j^2]} \tag{3.9}$$

It is clear that the fluxes j, p_i, and e must satisfy certain constraints for the Maxwellians to exist, because nonreal solutions are not admissible from a physical point of view.

The important conclusion for our purpose is this: For $j = p_2 = p_3 = e = 0$ and $p_1 > 0$, there is a manifold of Maxwellians with these fluxes j, p_i, and e. For $j > 0$ and p_1, p_2, p_3, and e such that there are at all Maxwellians having these fluxes, we observe that there are at most two.

This feature of one-dimensional flows is well known in the context of the theory of shock waves in an ideal fluid.[15] Actually, one of the two roots in Eq. (3.9) belongs to a subsonic, the other to a supersonic flow. From the definition of the Mach number M (based on the x component of the bulk velocity), we have

$$M^2 = \frac{u^2}{\frac{5}{3} P/\rho} = \frac{j^2}{\frac{5}{3}(p_1\rho - j^2)} \tag{3.10}$$

Here P denotes the pressure. An elementary study shows that the root with the plus sign from (3.9) satisfies $\rho \geqslant 8j^2/5p_1$, while the root with the minus sign satisfies $\rho \leqslant 8j^2/5p_1$, and hence $M \leqslant 1$ for the first root and $M \geqslant 1$ for the second.

From the results of a linearized analysis[11,19] we expect that the root for which $M > 1$ cannot be reached for positive j. We believe that the linearized analysis can be used to obtain a rigoros proof of this statement, but a detailed discussion of this point is beyond the scope of the present paper.

4. AUXILIARY RESULTS FOR BOUNDARY VALUE PROBLEMS IN A SLAB

In this section, we prepare for the main result by setting up and solving auxiliary problems in a finite interval $[0, d]$. The problems closely resemble problem (B) in ref. 12.

Choose $d > 0$ arbitrary but fixed. On $[0, d]$, we look for solutions of the following problems:

$$\xi_i \cdot \frac{d}{dx} f^i(x) = C^i(f)(x), \qquad x \in (0, d), \quad i = 1,..., n \tag{4.1}$$

$$\text{for} \quad \xi_i > 0, \qquad f^i(0) = \alpha_i \geqslant 0 \tag{4.2}$$

At the right boundary $x = d$, we impose one of the following two possible boundary conditions:

(i) For $\xi_i < 0$, choose numbers $\mu_i \geqslant 0$ arbitrary but fixed such that $\sum_i^- \mu_i = 1$, and a number $\kappa \in [0, 1)$, and prescribe $f^i(d)$ implicitly by

$$|\xi_i| \, f^i(d) = \kappa \mu_i j^+(d) \tag{4.3a}$$

(ii) or choose a number $t \geqslant 0$ which is smaller than or equal to the smallest positive ξ_i. Then prescribe $f^i(d)$ implicitly by

$$(t + |\xi_i|)\, f^i(d) = \mu_i[j^+(f)(d) - t\rho^+(f)(d)] \quad \text{if} \quad \xi_i < 0 \tag{4.3b}$$

There are many other boundary conditions that we could use for our purposes. Let us explain the virtues of (4.3a) and (4.3b), assuming that we have a nonnegative bounded solution for each problem (shortly, we will show that such solutions exist).

Condition (4.3a) enables us to calculate

$$\begin{aligned} j(x) &= j(d) = j^+(d) - j^-(d) \\ &= j^+(d) - \sum_i{}^- |\xi_i|\, f^i(d) \\ &= j^+(d) - \kappa \sum_i{}^- \mu_i j^+(d) = (1-\kappa)\, j^+(d) \end{aligned}$$

Suppose that $c_1 > 0$ is a lower bound and $c_2 > 0$ is an upper bound for all the $|\xi_i|$. This allows us to estimate

$$j^+(d) \geqslant c_1 \rho^+(d) \geqslant c_1 \rho(d)$$

Using the invariance of p_1, we arrive at

$$\begin{aligned} \rho(d) &= \sum_i (1/\xi_i)^2\, \xi_i^2 f^i(d) \geqslant c_2^{-2} p_1(d) \\ &= c_2^{-2} p_1(0) \geqslant c_2^{-2} p_1^+(0) \end{aligned} \tag{4.4}$$

Summarizing, (4.3a) guarantees that we have a uniform lower bound on $j(x)$:

$$j(x) \geqslant (1-\kappa)\, c_1 c_2^{-2} p_1^+(0) \geqslant 0$$

The condition (4.3b) is such that

$$j(f)(d) = t\rho(f)(d)$$

and from (4.4), we again get a uniform lower bound on $j(x)$ for positive t. Finally, we remark that condition (4.3b) for $t = 0$ enforces zero mass flux for any solution of (4.1)–(4.3b).

We now turn to the existence question.

Theorem 4.1. The problems (4.1)–(4.3a), (4.3b) have bounded, nonnegative, and continuous solutions. These solutions are bounded independent of d.

Proof. The same proof, already given in ref. 12, applies to both boundary conditions. For $\gamma > 0$, let

$$C_\gamma^i(f) = C^i(f) + \gamma\rho(f)$$

The structure of $C^i(f)$ is such that we can choose $\gamma > 0$ so large that C_γ^i maps $(C_{b,+})^n$ continuously into itself. Once γ is chosen, we keep it fixed. As in ref. 12, we define a mapping T: $(C_b)^n \to (C_{b,+})^n$ as follows. Take $g \in (C_b)^n$ and solve the problem

$$\xi_i \cdot \frac{d}{dx} f^i + \gamma f^i \rho(|g|) = C_\gamma^i(|g|) \tag{4.5}$$

[here $|g| = (|g^1|, |g^2|,..., |g^n|)$ with the boundary conditions (4.2) and (4.3a) or (4.3b)]. It is easy to see that this problem has a unique solution (see also ref. 12) and that this solution depends continuously on g. Hence, the formula

$$f = Tg$$

defines a continuous operator on $(C_b)^n$. The restriction of T to bounded sets is compact, because the derivatives of every solution of (4.5), (4.2), and (4.3a) or (4.3b) are uniformly bounded if g is from a bounded set.

We now use a theorem due to Schaefer[20,21] to conclude that T has a fixed point.

Suppose that $f = \lambda Tf$ for some λ, $0 < \lambda < 1$. Then, of course, $f \in (C_{b,+})^n$ and

$$\xi_i \cdot \frac{d}{dx} f^i + \gamma f^i \rho(f) = \lambda C^i(f) \tag{4.6}$$

$$f^i(0) = \lambda\alpha_i \qquad \text{if} \quad \xi_i > 0 \tag{4.7}$$

and, for either of the boundary conditions (4.3a) or (4.3b),

$$j(f)(d) \geqslant 0 \tag{4.8}$$

The definition of $C_\gamma^i(f)$ and the conservation equations (2.5) and (2.6) give

$$dj/dx = (\lambda - 1)\,\gamma\rho^2 \tag{4.9}$$

$$d\rho_1/dx = (\lambda - 1)\,\gamma j\rho \tag{4.10}$$

From the inequalities $\rho^2 \geqslant 0$ and $\lambda < 1$ and (4.9) we see that $j(x)$ (which is of course not constant for $\lambda \neq 1$) is non-increasing. Hence,

$$j(x) \geqslant j(d) \geqslant 0$$

From (4.10), it follows that p_1 is nonincreasing, i.e.,

$$\begin{aligned} p_1(x) &\leqslant p_1(0) = p_1^+(0) + p_1^-(0) \\ &\leqslant p_1^+(0) + c_2 j^-(0) \end{aligned} \tag{4.11}$$

Since $j(x)$ is nonnegative, we also have

$$0 \leqslant j(0) = j^+(0) - j^-(0)$$

so that $j^-(0) \leqslant j^+(0)$. This and (4.11) imply

$$p_1(x) \leqslant 2c_2 j^+(0) \tag{4.12}$$

Condition (4.12) is an *a priori* bound on all solutions of $f = \lambda Tf$. The Schaefer theorem then implies that $f = \lambda Tf$ has a solution for $\lambda = 1$. This is the solution we were looking for. That this solution is bounded independently of d is an immediate consequence of (4.12).

Remark. The theorem makes no statement about uniqueness. We know that the solution is unique for sufficiently small d,[12] but this uniqueness may be lost for large d.

5. THE MAIN THEOREM

We briefly remind the reader of the assumptions made for the models under consideration. They are (1) that all $\xi_i \neq 0$ and (2) that the mass, momentum, and energy fluxes are invariants. M will again denote the Maxwellian manifold associated with certain fixed values of the invariants. Given these assumptions, we prove the following result:

Theorem 5.1. There are bounded nonnegative solutions f of (2.1), (2.3) such that $\lim_{x \to \infty} \operatorname{dist}(f(x), M) = 0$, where M is the Maxwellian manifold associated with the invariants of f. If there are only finitely many points in M, then there is a Maxwellian in M such that $\lim_{x \to \infty} f(x) = m$.

Proof. Choose either of the problems (4.1)–(4.3a), or (4.3b). Then choose a sequence (d_n) of values of d increasing to infinity, and let (f_n) be the sequence of solutions for the d_n given by Theorem 4.1. For convenience, we extend f_n^i as the constant $f_n^i(d_n)$ for $x > d_n$.

By Theorem 4.1, the family (f_n) is bounded, and because of the boundedness of $(d/dx) f_n^i$, it is actually equicontinuous. By using a standard diagonal trick, we can extract a subsequence (f_{n_k}) which converges uniformly on every bounded interval and pointwise on $[0, \infty)$. The limit $f = (f^i)$ is a solution of (2.1), and satisfies the boundary condition at $x=0$ and $j(f)\geqslant 0$. It remains to show that f approaches the Maxwellian manifold as $x\to\infty$. To this end, we take advantage of the H-theorem.

The function f satisfies $(d/dx)\,\tilde{H}[f](x)\leqslant 0$. The function f is bounded because all the functions f_n are uniformly bounded, and so $\tilde{H}[f](x)$ is bounded below. Hence $\tilde{H}[f](\infty):=\lim_{x\to\infty}\tilde{H}[f](x)$ exists, and in fact

$$\tilde{H}[f](0)-\tilde{H}[f](\infty)=-\int_0^\infty\left\{\frac{d}{dx}\,\tilde{H}[f](x)\right\}dx$$

Therefore, $(d/dx)\,\tilde{H}[f](x)$ is L^1-integrable, and so we can find a sequence $x_k\to\infty$ such that $(d/dx)\,\tilde{H}[f](x_k)\to 0$ as $k\to\infty$. The sequence $(f(x_k))$ is bounded, so the Bolzano–Weierstrass theorem allows us to extract a subsequence [which we again denote by (x_k)] such that $\lim_{k\to\infty}f(x_k)=g$ exists. Now recall that

$$\frac{d}{dx}\,\tilde{H}[f](x_k)=\sum_{i,j,k,l}A_{ij}^{kl}(f_kf_l-f_if_j)\ln\frac{f_kf_l}{f_if_j}(x_k) \tag{5.1}$$

Since

$$(f_kf_l-f_if_j)\ln\frac{f_kf_l}{f_if_j}(x_k)\leqslant 0$$

the limit of this expression as $k\to\infty$ must exist and vanish if $A_{ij}^{kl}\neq 0$. We conclude that

$$\begin{aligned}\lim_{k\to\infty}\;&(f_kf_l-f_if_j)\ln\frac{f_kf_l}{f_if_j}(x_k)\\&=(g_kg_l-g_ig_j)\lim_{k\to\infty}\ln\frac{f_kf_l}{f_if_j}(x_k)=0\end{aligned}$$

whenever $A_{ij}^{kl}\neq 0$, and this is only possible when g is Maxwellian. Clearly, g has the invariant fluxes of f.

Given that M is the Maxwellian manifold associated with the fluxes of f, it remains to show that

$$\lim_{k\to\infty}\operatorname{dist}(f(y_k),M)=0$$

for every other sequence (y_k) with $y_k\to\infty$. Suppose that this is false; we can then find numbers $\varepsilon_1>0$ and $\delta>0$ and an increasing sequence (y_k) with $|y_{k+1}-y_k|>\varepsilon_1$ and $\operatorname{dist}(f(y_k),M)>\delta$ for all k.

Because the derivative of f is uniformly bounded, there is an $\varepsilon_2 > 0$, $\varepsilon_2 < \varepsilon_1/2$, and a sequence of intervals Δ_k of length ε_2 centered at y_k such that $\operatorname{dist}(f(x), M) > \delta/2$ for all $x \in \Delta_k$ and all k.

Let $\Delta_k = [a_k, b_k]$ and consider the entropy flux into Δ_k: With $h(y) = y \ln y$ for $y > 0$, $h(y) = 0$ for $y = 0$, define

$$\phi_{\text{in}}(\Delta_k) = \sum{}^{+} \xi_i h(a_k) + \sum{}^{-} |\xi_i|\, h(b_k)$$

$$\phi_{\text{out}}(\Delta_k) = \sum{}^{+} \xi_i h(b_k) + \sum{}^{-} |\xi_i|\, h(a_k)$$

$$\phi(\Delta_k) = \phi_{\text{in}}(\Delta_k) - \phi_{\text{out}}(\Delta_k)$$

The decrease of $\tilde{H}[f](x)$ easily implies that $\phi(\Delta_k) \geqslant 0$. To finish the argument, we the use the following result:

Lemma 5.2. The assertion $\operatorname{dist}(f(x), M) > \delta/2$ for $x \in \Delta_k$ implies that there is a constant $\lambda > 0$ such that

$$\phi(\Delta_k) \geqslant \lambda \tag{5.2}$$

for all k.

Before we prove the lemma, we finish the proof of the theorem.

Observe that $\phi(\Delta_k) = \tilde{H}[f](a_k) - \tilde{H}[f](b_k)$. Since $\tilde{H}[f](x)$ is decreasing and $\int_0^\infty (d/dx)\, \tilde{H}[f](x) x < \infty$, it follows that

$$\sum_k \phi(\Delta_k) < \infty \tag{5.3}$$

(5.3) contradicts (5.2), and our proof is complete.

If the Maxwellian manifold M consists of at most finitely many points, $\operatorname{dist}(f(x), M)$ can only converge to zero as $x \to \infty$ if f actually converges to a Maxwellian.

Proof of Lemma 5.2. The proof is again by contradiction. Assuming that the assertion of the lemma is false, we can find a subsequence of the intervals (Δ_k) [again denoted by (Δ_k)] such that $\lim_{k \to \infty} \phi(\Delta_k) = 0$. Since

$$\phi(\Delta_k) = -\int_{a_k}^{b_k} \frac{d}{dx} \tilde{H}[f](x)\, dx$$

there must be numbers $z_k \in \Delta_k$ such that $(d/dx)\, \tilde{H}[f](z_k) \to 0$ as $k \to \infty$. As before, this implies that $f(z_k)$ approaches the Maxwellian manifold, and this is in contradiction to the hypothesis of the lemma.

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